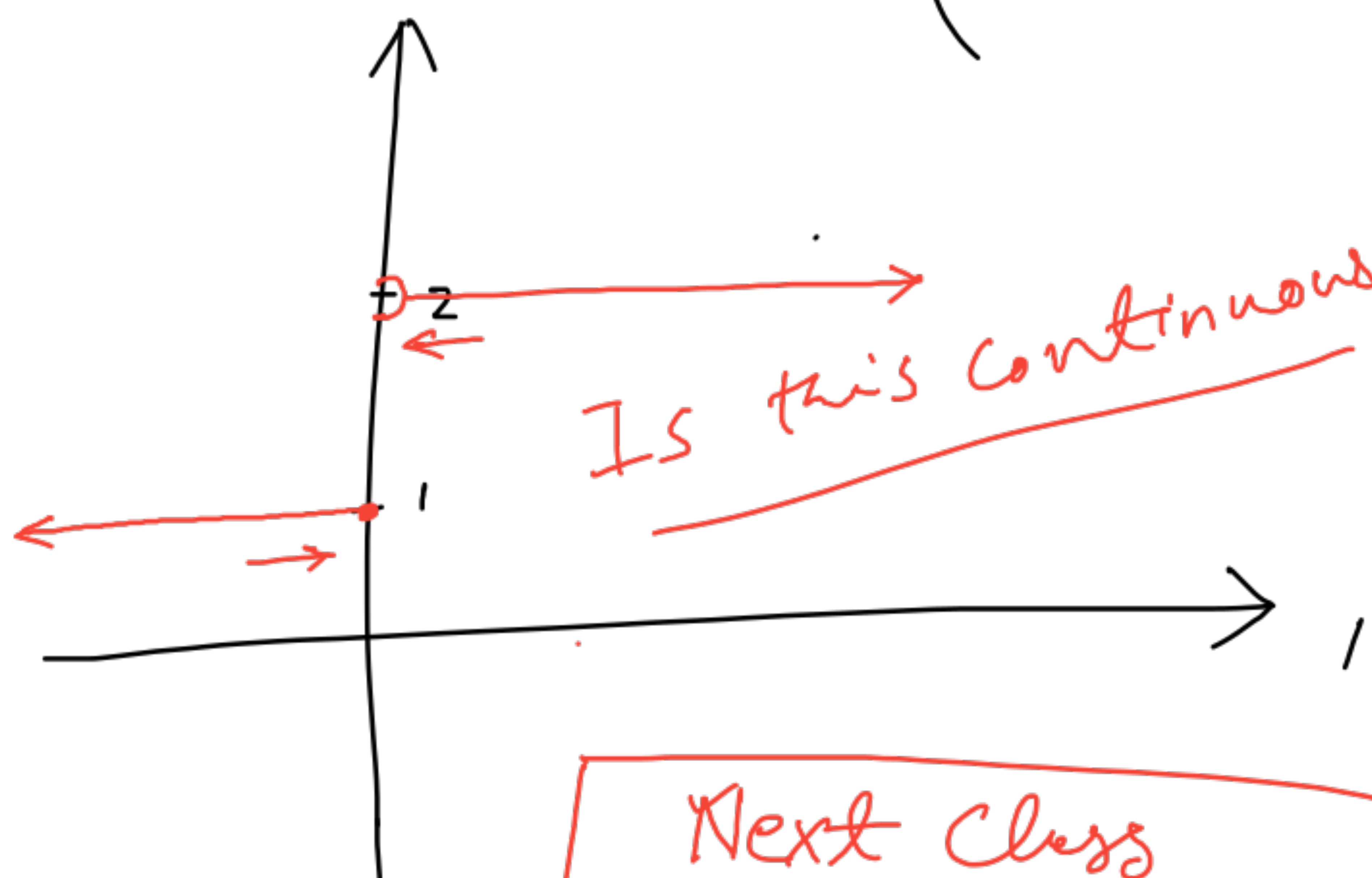
LIVE 5.00PM Continuous Function

Let $f(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ 2 & \text{if } x > 1 \end{cases}$



Is this continuous? LHL $= +1$

R.H.L. $= 2$

Next Class

Short techineques on Cont. Function

16th April 13.00hr.

Let f be a real function on a subset of the real numbers and let c be a point in the domain of f. Then f is continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

More elaborately, if the left hand limit, right hand limit and the value of the function at $x = c$ exist and are equal to each other, i.e.,

$$\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x)$$

then f is said to be continuous at $x = c$.

OK

More clearly

$$\text{L.H.L.} = \lim_{x \to c^-} f(x) = \text{RHL} = \lim_{x \to c^+} f(x) = f(c)$$

Then $f(x)$ is continuous.

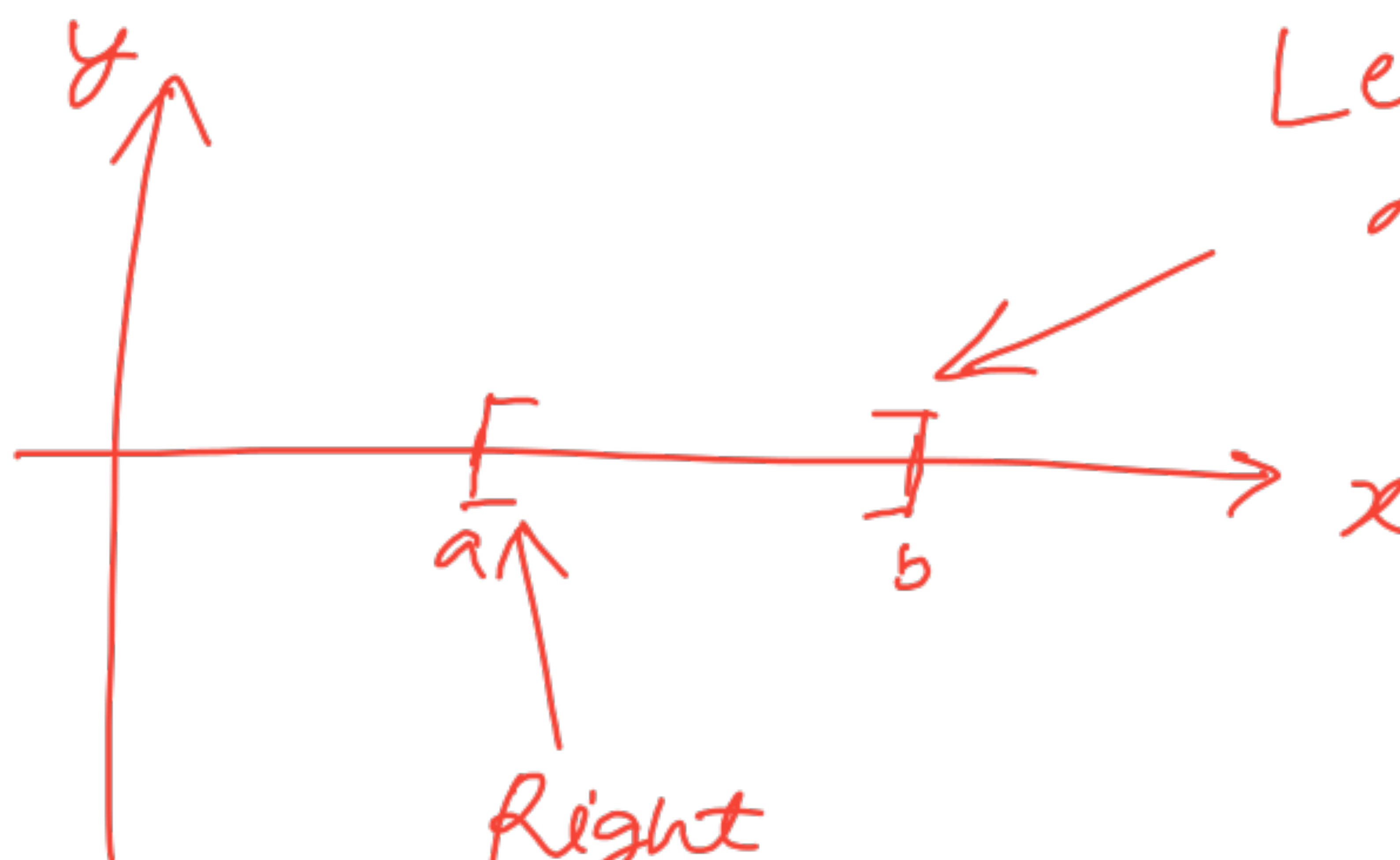
Continuity in an interval

(i) f is said to be continuous in an open interval (a, b) if it is continuous at every point in this interval.

(ii) f is said to be continuous in the closed interval $[a, b]$ if

- f is continuous in (a, b)
- $\lim_{x \to a^+} f(x) = f(a)$
- $\lim_{x \to b^-} f(x) = f(b)$

Left Cont. at Right Point



Right Continuous at Left Point

Some Common Continuous Functions

Function $f(x)$	Interval in which f is continuous
1. The constant function, i.e. $f(x) = c$	
2. The identity function, i.e. $f(x) = x$	**R**
3. The polynomial function, i.e. $f(x) = a_0 x^n + a_1 x^{n-1} + ... + a_{n-1} x + a_n$	
4. $\lvert x - a \rvert$	$(-\infty, \infty)$
5. x^{-n}, n is a positive integer	$(-\infty, \infty) - \{0\}$
6. $p(x) / q(x)$, where $p(x)$ and $q(x)$ are polynomials in x	$\mathbf{R} - \{x : q(x) = 0\}$
7. $\sin x$, $\cos x$	**R**
8. $\tan x$, $\sec x$	$\mathbf{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\}$
9. $\cot x$, $\operatorname{cosec} x$	$\mathbf{R} - \{(n\pi : n \in \mathbf{Z}\}$

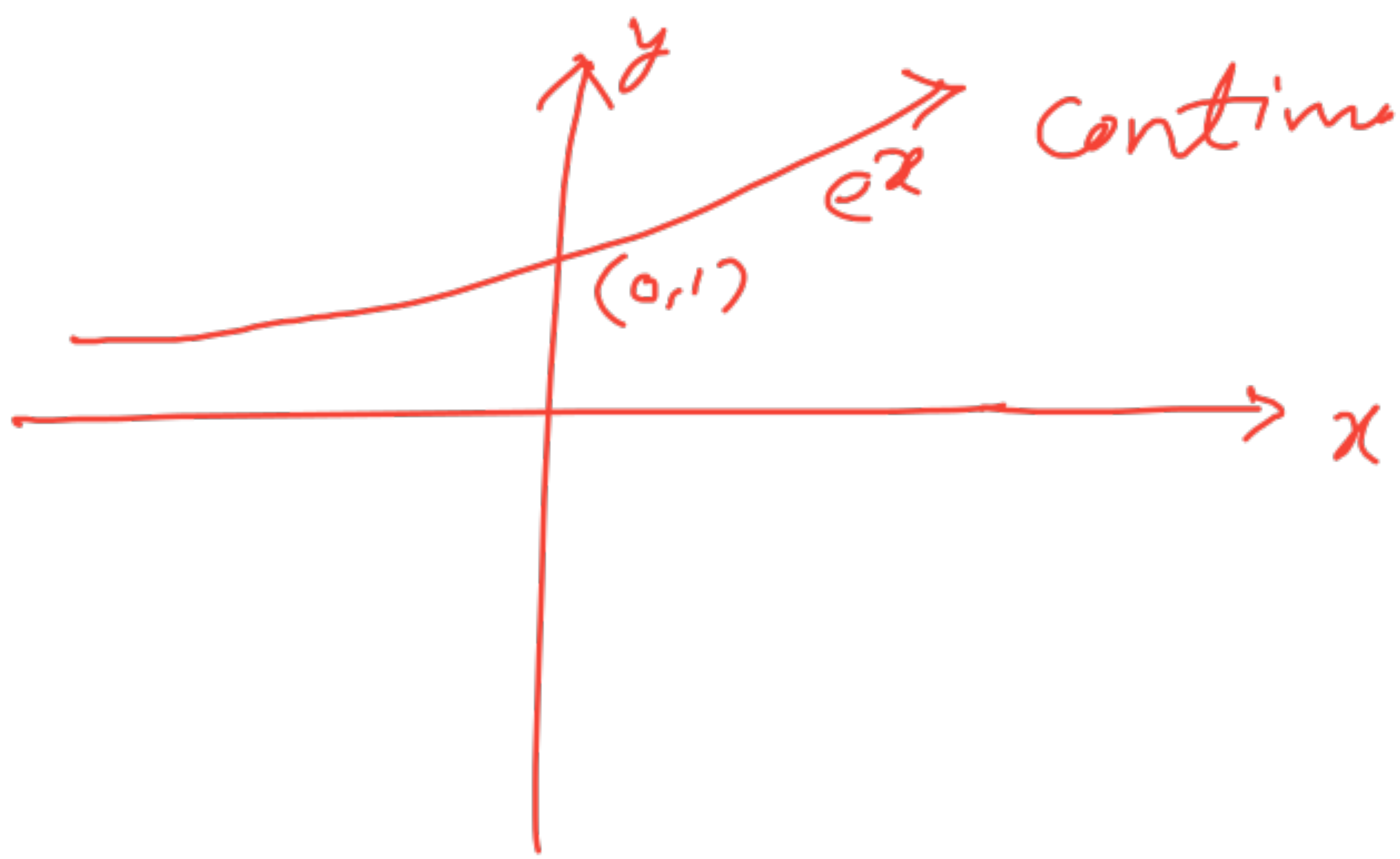
Trigonometric function

Check Domain of definition of func.

9 Points Polynomial are always continuous.

10. e^x	**R**
11. $\log x$	$(0, \infty)$
12. The inverse trigonometric functions, i.e., $\sin^{-1} x$, $\cos^{-1} x$ etc.	In their respective domains

e^x



continu..

Some Spl. Function

SIGNUM FUNCTION *This function is defined by* $f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0. \end{cases}$

Thus, we have $f(x) = \begin{cases} 1, & \text{when } x > 0 \\ 0, & \text{when } x = 0 \\ -1, & \text{when } x < 0. \end{cases}$

Clearly, its domain is R and range $= \{-1, 0, 1\}$.

Continuus ??

Home work

STEP FUNCTION OR THE GREATEST INTEGER FUNCTION

If $x \in R$ then $[x]$ is defined as the greatest integer not exceeding x.

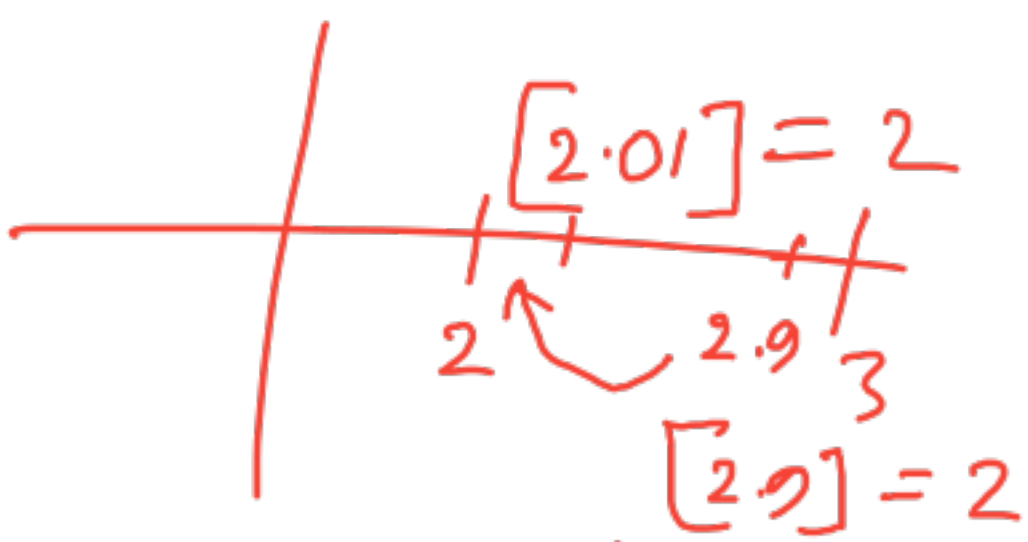
For example, we have

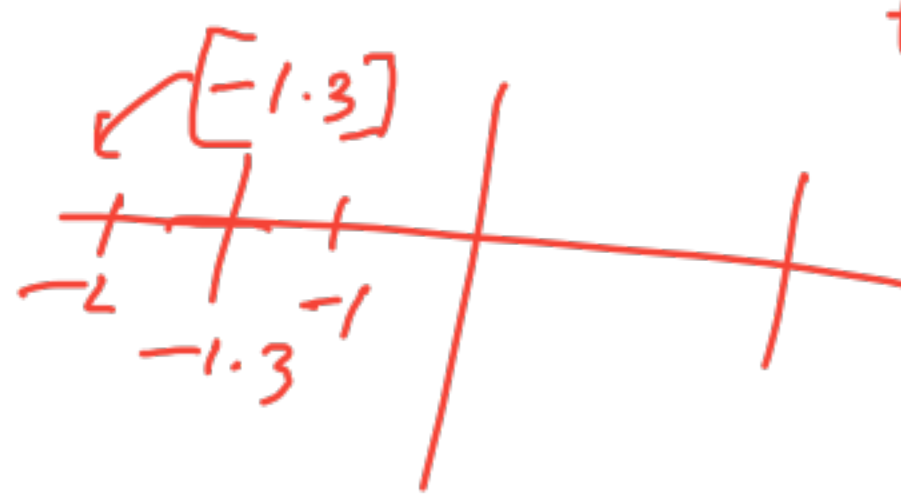
$[2.01] = 2$; $[2.9] = 2$; $[-1.3] = -2$; $[3] = 3$ and $[-1] = -1$, etc.

$[x] \rightarrow$ takes left integer value

Step Function | Box Function | GIF function

Gretest Integer Function





8. SMALLEST INTEGER FUNCTION (OR CEILING FUNCTION) *For any real number x, we define $\lceil x \rceil$ as the smallest integer greater than or equal to x.*

For example,

$$\lceil 6.3 \rceil = 7,\ \lceil 7.01 \rceil = 8,\ \lceil -6.1 \rceil = -6,\ \lceil -2.9 \rceil = -2,\ \lceil -3 \rceil = -3,\ \lceil 5 \rceil = 5.$$

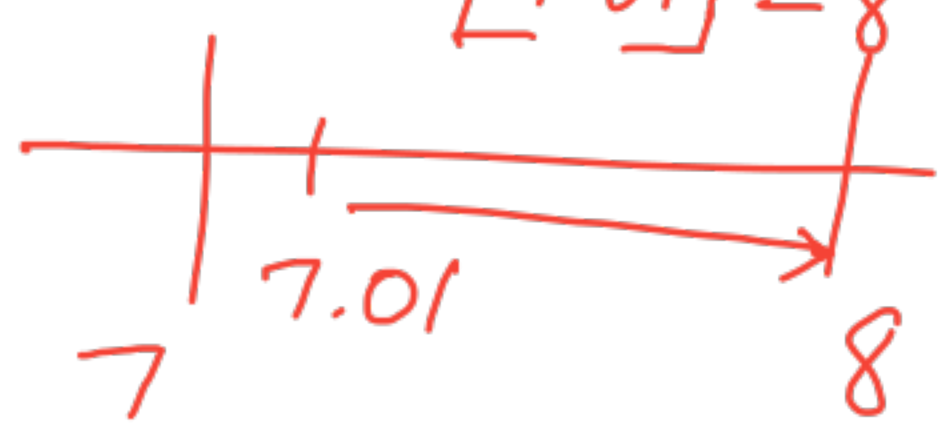
The function $f : R \in R : f(x) = \lceil x \rceil,\ x \in R$ is called the smallest integer function or ceiling function.

Clearly, domain$(f) = R$ and range$(f) = I$.

Ceiling Function

lakar Right side integer vali

$\lfloor 7.01 \rfloor = 8$



9. POLYNOMIAL FUNCTION *A function of the form*

$$p(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n,$$

where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ *are real numbers,* $a_0 \neq 0$ *and n is a non-negative integer, is called a polynomial function of degree n.*

Polynomials of degree 1, 2, 3 and 4 are respectively called *linear, quadratic, cubic* and *biquadratic* polynomials.

Thus, (i) $f(x) = ax + b,\ a \neq 0$, is a linear polynomial.

(ii) $f(x) = ax^2 + bx + c,\ a \neq 0$, is a quadratic polynomial.

(iii) $f(x) = ax^3 + bx^2 + cx + d,\ a \neq 0$, is a cubic polynomial.

$= a \neq 0$

Polynomial function of degree n

$$p(x) = \sum_{n=0}^{n} a_n x^n$$

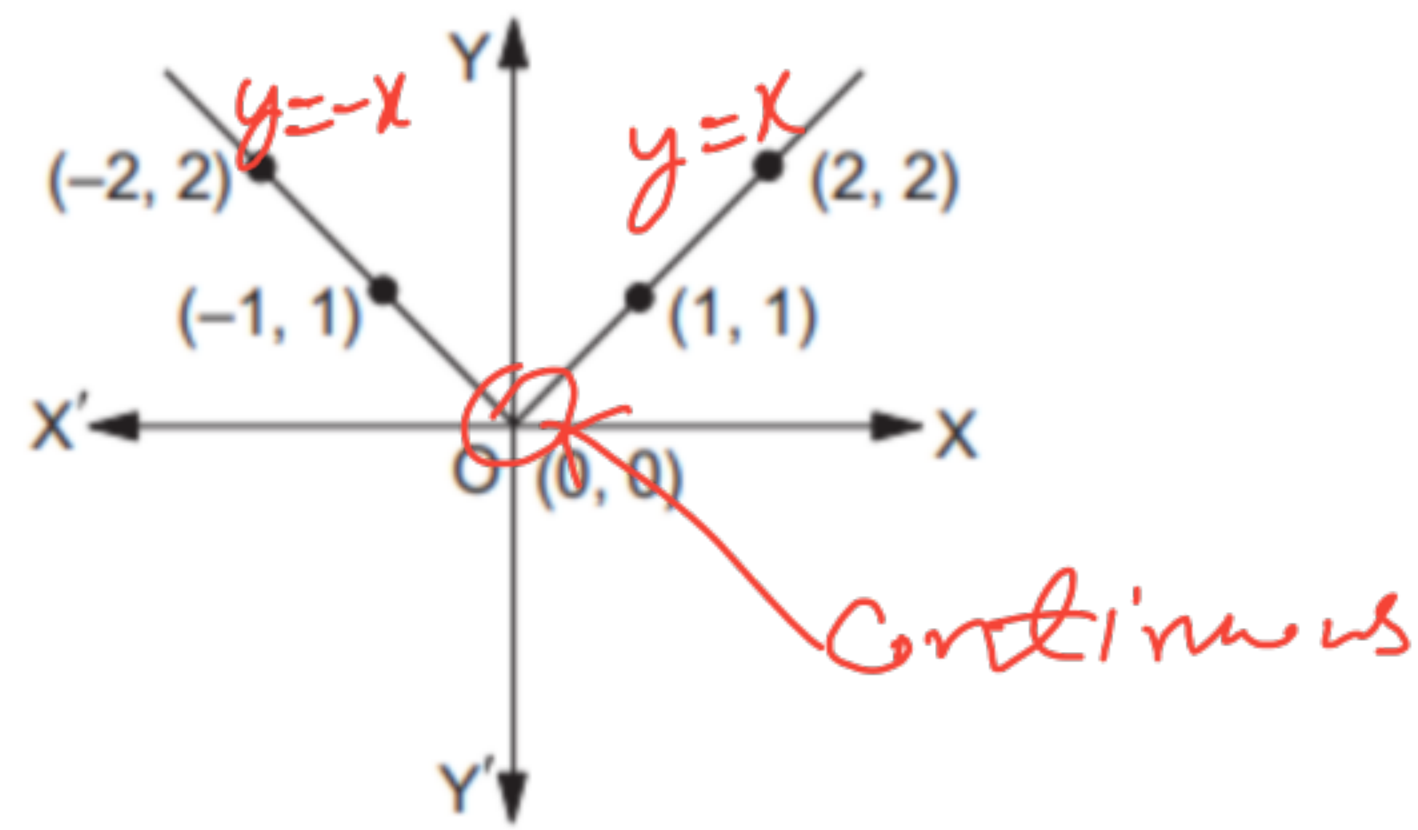
$a_n \neq 0$

Draw the graph of the modulus function $f(x) = |x|$.

$$f(x) = |x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$$

Some of the points on the graph are $(0, 0), (-1, 1), (-2, 2), (1, 1), (2, 2)$, etc.

Joining these points, we get the required graph.

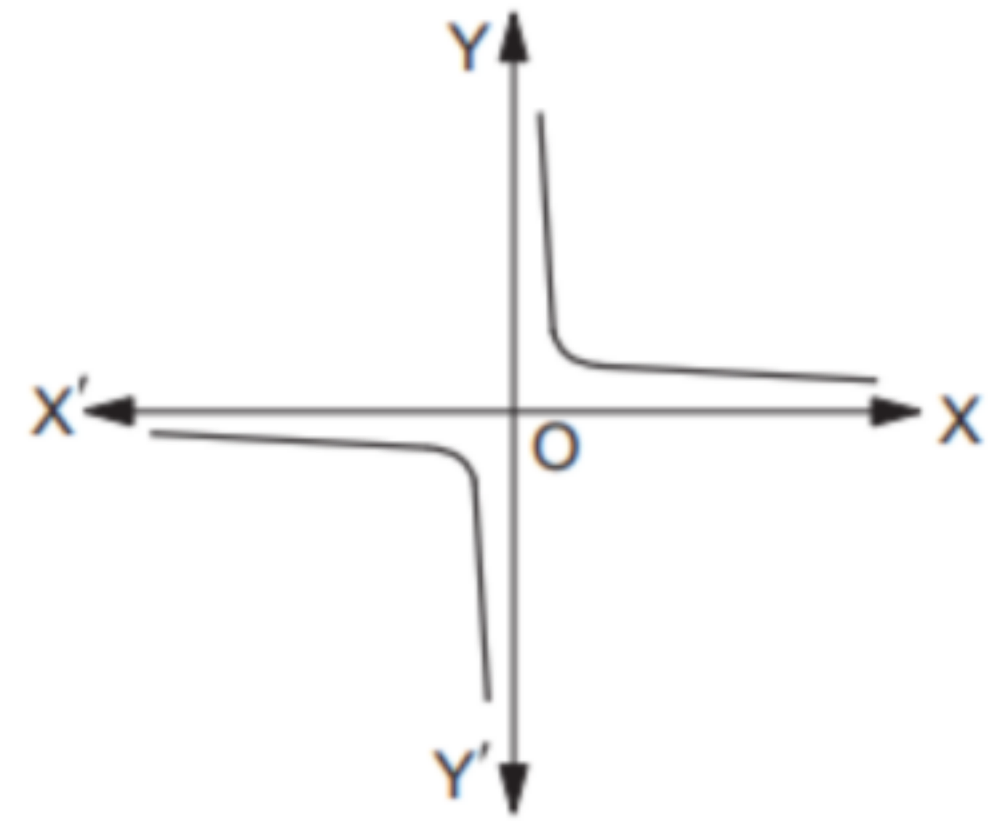


$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$$

Draw the graph of the reciprocal function $f(x) = \dfrac{1}{x}$.

Clearly, $f(x) = \dfrac{1}{x}$ is not defined at $x = 0$. Some points on the graph are $(1, 1)$, $(-1, -1)$, $(1/2, 2)$, $(2, 1/2)$, $(-1/2, -2)$, $(-2, -1/2)$, $(1/3, 3)$, $(-1/3, -3)$, $(3, 1/3)$, etc.

Joining these points, we get the required graph. Since $f(x) = \dfrac{1}{x}$ is an odd function, it is symmetrical about the origin.



Reciprocal function

Continuous at $R - \{0\}$

0 pe undefined

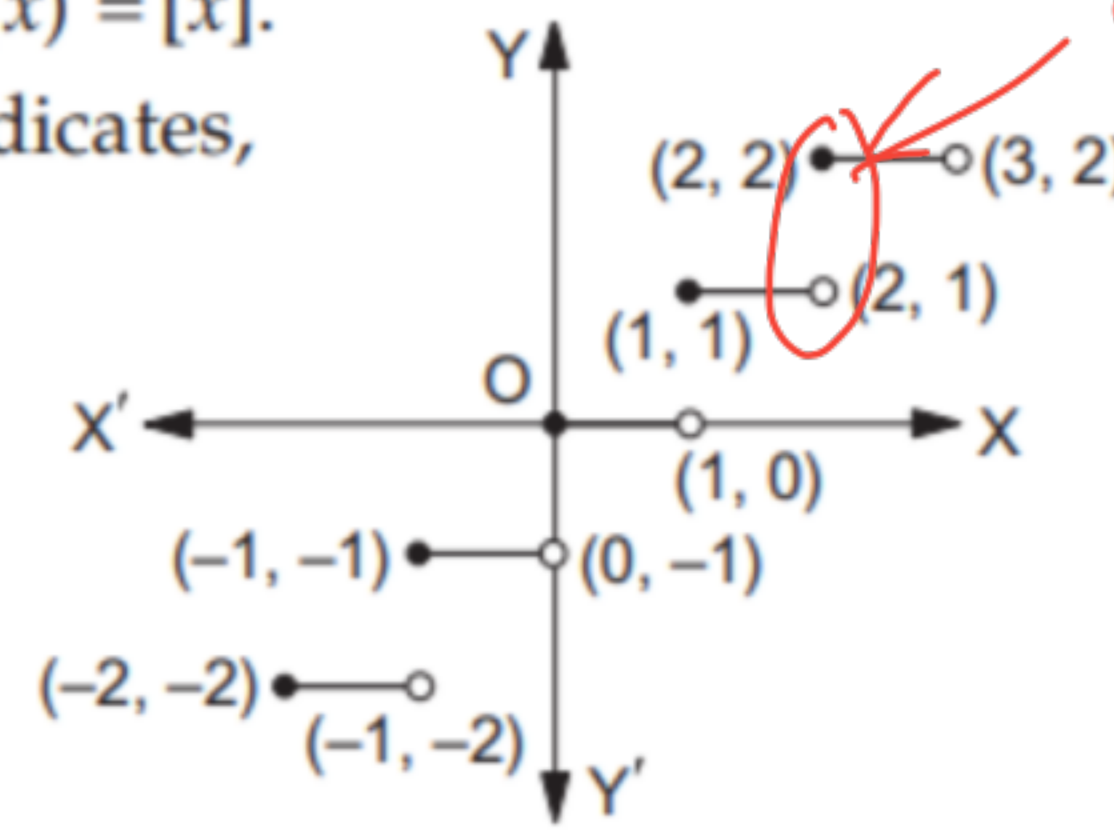
Draw the graph of the step function $f(x) = [x]$.

As the definition of the function indicates,

for all x such that $-2 \le x < -1$,
we have $f(x) = -2$;

for all x such that $-1 \le x < 0$,
we have $f(x) = -1$;

for all x such that $0 \le x < 1$,
we have $f(x) = 0$;



Continuty Breaks

for all x such that $1 \le x < 2$, we have $f(x) = 1$, and so on,

$$\text{i.e., } f(x) = \begin{cases} -2 \text{ when } x \in [-2, -1[\\ -1 \text{ when } x \in [-1, \ 0[\\ \ \ 0 \text{ when } x \in [0, \ 1[\\ \ \ 1 \text{ when } x \in [1, \ 2[\\ \ \ \text{and so on.} \end{cases}$$

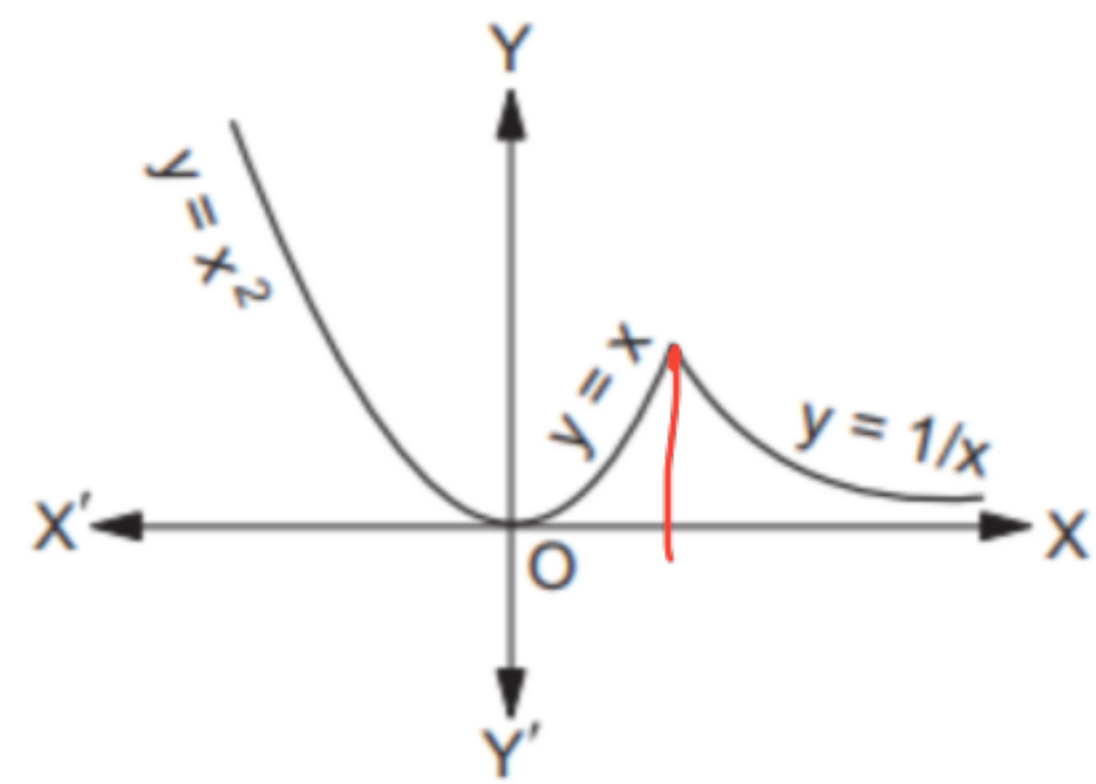
Clearly, the function jumps at the points (–1, –2), (0, –1), (1, 0), (2, 1), etc.

In other words, *the given function is discontinuous at each integral value of* x.

Box function clearly not continuous in $\mathbb{R}$.

Draw the graph of the function $f(x) = \begin{cases} x^2, & \text{when } x < 0 \\ x, & \text{when } 0 \le x \le 1 \\ 1/x, & \text{when } 1 \le x < \infty. \end{cases}$

Here, the graph consists of three parts. Some of the points of the graph are $(-3, 9)$, $(-2, 4)$, $(-1, 1)$, $(0, 0)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{3}{4}, \frac{3}{4}\right)$, $(1, 1)$, $\left(2, \frac{1}{2}\right)$, $\left(3, \frac{1}{3}\right)$, etc.



And, the graph may now be drawn, as shown in the adjoining figure.

Conclusion: Graphical techinique

Next class on this topic

Short Trick

To check continuity

Discuss the continuity of the function $f(x)$ at $x = 0$, if

$$f(x) = \begin{cases} 2x - 1, & x < 0 \\ 2x + 1, & x \geq 0. \end{cases}$$

[CBSE 2002]

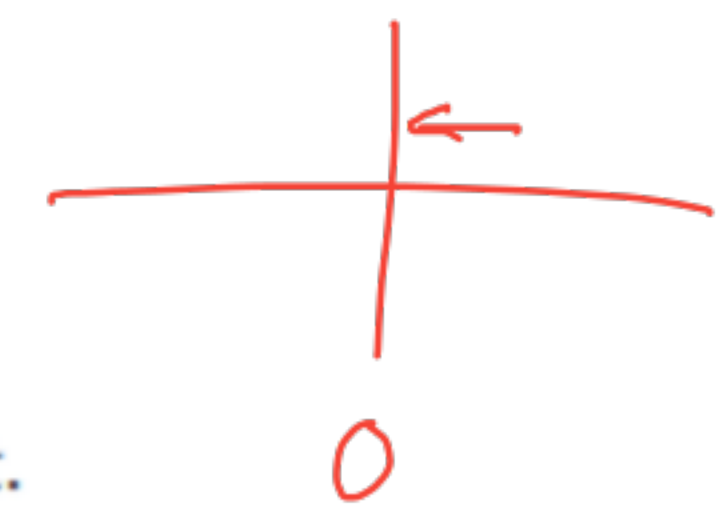
Clearly, $f(0) = (2 \times 0 + 1) = 1.$

$$\boxed{\lim_{x \to 0+} f(x)} = \lim_{h \to 0} f(0 + h)$$

$$= \lim_{h \to 0} [2(0 + h) + 1] = \lim_{h \to 0} (2h + 1) = 1.$$

$$\boxed{\lim_{x \to 0-} f(x)} = \lim_{h \to 0} f(0 - h)$$

$$= \lim_{h \to 0} [2(0 - h) - 1)] = \lim_{h \to 0} (-2h - 1) = -1.$$



Thus, $\lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x)$ and therefore, $\lim_{x \to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

Right Continuous ? ✓

Left continuous ? ✗

exercise

Show that the function $f(x) = \begin{cases} \left(\dfrac{e^{1/x}-1}{e^{1/x}+1}\right), & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$

is discontinuous at $x = 0$.

Now, $\lim\limits_{x \to 0} f(x) = \lim\limits_{h \to 0} f(0+h) = \lim\limits_{h \to 0} f(h) = \lim\limits_{h \to 0} \left(\dfrac{e^{1/h}-1}{e^{1/h}+1}\right)$ ← techinque to Solve limit

$$= \lim_{h \to 0} \frac{e^{1/h}\left(1 - \dfrac{1}{e^{1/h}}\right)}{e^{1/h}\left(1 + \dfrac{1}{e^{1/h}}\right)} = \lim_{h \to 0} \frac{\left(1 - \dfrac{1}{e^{1/h}}\right)}{\left(1 + \dfrac{1}{e^{1/h}}\right)} = 1.$$

And, $\lim\limits_{x \to 0} f(x) = \lim\limits_{h \to 0} f(0-h) = \lim\limits_{h \to 0} f(-h) = \lim\limits_{h \to 0} \left(\dfrac{e^{-1/h}-1}{e^{-1/h}+1}\right)$

$$= \lim_{h \to 0} \frac{\left(\dfrac{1}{e^{1/h}} - 1\right)}{\left(\dfrac{1}{e^{1/h}} + 1\right)} = -1.$$

Short Tech.

L'Hospital Rule

Nex Class

Thus, $\lim\limits_{x \to 0+} f(x) \neq \lim\limits_{x \to 0-} f(x)$, and therefore, $\lim\limits_{x \to 0} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$.

Solution of Rajasthan Board

(b) $f(x) = \begin{cases} \frac{e^{-x}}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$$\underset{x\to 0}{Lt} \frac{e^{-x}}{x} = \underset{x\to 0}{Lt} \frac{(-1)e^{-x}}{1} = -1 \neq 0 = f(0)$$

Result : $f(x)$ is not Continuous.

(c) $f(x) = \begin{cases} 1+x, & x \leq 3 \\ 7-x, & x > 3 \end{cases}$

$$\underset{x\to 3-}{Lt} f(x) = \underset{x\to 3-}{Lt} 1+x = 1+3 = 4$$

$$\underset{x\to 3+}{Lt} f(x) = \underset{x\to 3+}{Lt} 7-x = 7-3 = 4$$

Also, $f(3) = (1+x)\big|_{x=3} = 1+3 = 4$

So, $\underset{x\to 3-}{Lt} f(x) = \underset{x\to 3+}{Lt} f(x) = f(3)$

Result: $f(x)$ Continuous at $x=3$

(d) $f(x) = \begin{cases} \sin x & -\frac{\pi}{2} < x \leq 0 \\ \tan x & 0 < x < \frac{\pi}{2} \end{cases}$

$f(0) = \sin 0 = 0$

$\underset{x \to 0-}{Lt} f(x) = \underset{x \to 0-}{Lt} \sin x = \sin 0 = 0$

$\underset{x \to 0+}{Lt} f(x) = \underset{x \to 0+}{Lt} \tan x = \tan 0 = 0$

$\therefore f(0) = \underset{x \to 0}{Lt} f(x)$, f is continuous at $x=0$

(e) $f(x) = \begin{cases} \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

$f(0) = 0$

$\underset{x \to 0}{Lt} \cos\left(\frac{1}{x}\right)$ does not exist as

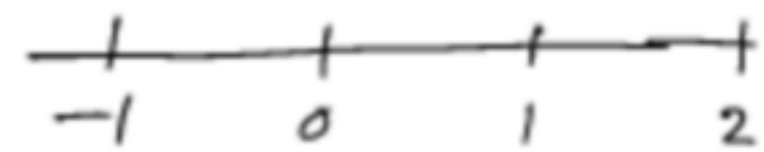
$\cos\left(\frac{1}{x}\right)$ is oscillatory at $x \to 0$

within $[-1, 1]$

$\Rightarrow f(x)$ is not continuous.

(4) $$f(x) = \begin{cases} -x^2 & , -1 \le x < 0 \\ 4x-3 & , 0 < x \le 1 \\ 5x^2-4x & , 1 < x \le 2 \end{cases}$$

$$\underset{x\to(-1)+}{Lt} f(x) = \underset{x\to(-1)^+}{Lt} (-x^2)$$



$$= -(-1)^2$$

$$= -1 = f(-1)$$

f is $_{\wedge}^{\text{Right}}$ Cont. at $x = -1$

$$\left.\begin{aligned} \underset{x\to 0-}{Lt} f(x) &= \underset{x\to 0-}{Lt} (-x^2) = 0 \\ \underset{x\to 0+}{Lt} f(x) &= \underset{x\to 0+}{Lt} 4x-3 = -3 \end{aligned}\right\} \Rightarrow f \text{ is not Cont. at } x=0.$$

$$\underset{x\to 1+}{Lt} f(x) = \underset{x\to 1+}{Lt} 5x^2-4x = 5.1^2-4.1$$
$$= 1$$

$$\underset{x\to 1-}{Lt} f(x) = \underset{x\to 1-}{Lt} 4x-3 = 4.1-3 = 1$$

$$f(1) = 4.1-3 = 1 = \underset{x\to 1}{Lt} f(x)$$

$\Rightarrow f$ is Cont. at $x = 1$.

$$\underset{x\to 2-}{Lt} f(x) = \underset{x\to 2-}{Lt} 5x^2-4x = 5.2^2-4.2$$